

# Three ignored densities, frame-independent thermodynamics, and broken Galilean symmetry

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A system's invariance under Galilean transformation implies three locally conserved densities. Including them as variables, the thermodynamics is rendered explicitly frame independent, dissipative mass currents are shown to vanish, and spontaneously broken Galilean symmetry becomes a sensible concept in condensed systems. [S1063-651X(98)08411-6]

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## I. INTRODUCTION

The invariance under Galilean boosts of a closed system implies the conservation of

$$\mathbf{B} = M\mathbf{R}_0, \quad (1)$$

where  $M$  is the system's mass, and  $\mathbf{R}_0$  its center of mass coordinate at the time  $t=0$  [1]. Despite the appearance of an initial value that can always be set to zero by an appropriate choice of the coordinate,  $\mathbf{B}$  is an additive and locally conserved quantity, in complete analogy to energy, mass, momentum, and angular momentum. As these are all thermodynamic and hydrodynamic variables, it is a valid question why  $\mathbf{B}$  is never included. This paper shows the usefulness, even necessity, of including  $\mathbf{B}$ , summarized in the following list.

(i) The inclusion of  $\mathbf{B}$  as a variable is a necessary condition for the formulation of a frame-independent thermodynamic theory: Starting from the rest frame expression, say  $dE = TdS$ , it may seem all right to add  $\mathbf{V} \cdot d\mathbf{G}$  for a boosted system with the momentum  $\mathbf{G}$ , and add  $\mathbf{\Omega} \cdot d\mathbf{L}$  if it also rotates with the angular momentum  $\mathbf{L}$ , but this is not enough. As shown in Sec. II, this is only correct for frames in which  $\mathbf{\Omega} \parallel \mathbf{G}$ , while generally the energy  $E$  also depends on  $\mathbf{B}$ .

Before going on with the list, it is convenient to introduce a name for

$$\mathbf{B} = M(\mathbf{R} - \dot{\mathbf{R}}t) = \int d^3x(\varrho\mathbf{x} - \mathbf{g}t), \quad (2)$$

where  $\mathbf{R} = \mathbf{R}_0 + \dot{\mathbf{R}}t$  is the time-dependent center of mass coordinate, with a constant time derivative  $\dot{\mathbf{R}} = \mathbf{G}/M$ , while  $\varrho$  and  $\mathbf{g}$  denote the density of mass and momentum, respectively. Following Schwinger [2], we shall refer to  $\mathbf{B}$  as the "booster," and to  $\mathbf{b} = \varrho\mathbf{x} - \mathbf{g}t$  as the "booster density." Including the energy's dependence on the booster,  $dE = \dots + \mathbf{A} \cdot d\mathbf{B}$ , the conjugate variable  $\mathbf{A}$  will turn out to be  $\mathbf{\Omega} \times \dot{\mathbf{R}}$ .

(ii) One of the more direct results of the thermodynamic theory is the equilibrium conditions. Again, the correct derivation requires the inclusion of the booster, although this is never done. If one derives them following Landau and Lifshitz [3] by maximizing the entropy while holding constant

the energy, mass, momentum, and angular momentum, we find as equilibrium conditions constant temperature and chemical potential, with the velocity given as  $\mathbf{v} = \mathbf{V} + \mathbf{\Omega} \times \mathbf{x}$ , where  $\mathbf{V}$  and  $\mathbf{\Omega}$  are again constant. Due to the lack of a time-dependent term, however, this velocity is less general than the usual expression for a rigid body,  $\mathbf{v} = \dot{\mathbf{R}} + \mathbf{\Omega} \times (\mathbf{x} - \mathbf{R})$ , or equivalently

$$\mathbf{v} = (\dot{\mathbf{R}} - \mathbf{\Omega} \times \mathbf{R}_0) + \mathbf{\Omega} \times \mathbf{x} - (\mathbf{\Omega} \times \dot{\mathbf{R}})t. \quad (3)$$

(The difference is in the last term.) This is worrisome as there is no reason whatever why a system executing the general motion should not be in equilibrium. If, however, the booster is also held constant in the above calculation, one does arrive at the general expression, Eq. (3). At the same time, one finds that the chemical potential, instead of being constant, now satisfies  $\partial_t \nu + \mathbf{\nabla} \cdot \mu = 0$ . Reassuringly, this is exactly the expression the Navier-Stokes equation reduces to for vanishing entropy production, in equilibrium.

(iii) Local conservation of the mass density holds if the continuity equation  $\partial_t \varrho + \mathbf{\nabla} \cdot \mathbf{J} = 0$  is satisfied—irrespective of what form the mass current  $\mathbf{J}$  actually assumes. Usually, this form is taken as  $\mathbf{J} = \varrho\mathbf{v}$ , although it has never been properly deduced; rather, it is accepted as a statement of microscopic plausibility, or the summary of countless experiments. So no objection was, or could have been, raised, when Dzyaloshinskii and Volovik, in their classic paper [4], proposed inclusion of dissipative mass currents such as

$$\mathbf{J} - \varrho\mathbf{v} \sim \mathbf{\nabla} \mu. \quad (4)$$

(These dissipative terms result from a tempting, even natural, step to take when setting up the hydrodynamic equations.) On the other hand, there is a well-hidden footnote by Landau and Lifshitz [5] that purports to rule out this type of terms—but actually falls short of being ironclad if scrutinized: It invokes the center of mass motion to show that  $\int d^3x \mathbf{J} = \int d^3x \mathbf{g}$  must prevail, where the integration is to be taken over the volume of the system. For reasons of Galilean invariance, the momentum density is given as  $\mathbf{g} = \varrho\mathbf{v}$ . So a clear-cut proof of  $\mathbf{J} = \mathbf{g}$  would indeed serve as a sound argument for ruling out any dissipative mass currents. Unfortunately,  $\int d^3x \mathbf{J} = \int d^3x \mathbf{g}$  is less confining, and the reader is left wondering about terms that vanish only if integrated.

As will be discussed in Sec. III, taking the booster density  $\varrho\mathbf{x} - \mathbf{g}t$  as a locally conserved quantity that satisfies a conti-

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nity equation quickly leads to the result  $J = \mathbf{g}$ . The proof takes place in very much the same way as that deducing the symmetry of the stress tensor from the local conservation of angular momentum. (This result also has relativistic ramifications, as it shows that out of the two versions of relativistic hydrodynamics, by Eckart and by Landau-Lifshitz, only the latter is a proper generalization [6]. Any linear combination of both is also ruled out.)

(iv) The angular momentum frequently contains an extensive part  $\mathbf{s}$  that is usually referred to as its intrinsic contribution,

$$\mathbf{L} = \int d^3x (\mathbf{x} \times \mathbf{g} + \mathbf{s}). \quad (5)$$

One example of  $\mathbf{s}$  is the spin density. Due to the close relationship between  $\mathbf{L}$  and  $\mathbf{B}$ —the latter being, relativistically, part of the 4-tensor of the angular momentum density—it is certainly not farfetched to question whether a corresponding extensive contribution to the booster may exist,

$$\mathbf{B} = \int d^3x (\varrho \mathbf{x} - \mathbf{g}t + \mathbf{k}). \quad (6)$$

(Relativistically, the components of the 4-angular-momentum density mix under a Lorentz boost. And  $\mathbf{s}$ , the extensive contribution of the angular momentum in one frame, will lead to  $\mathbf{k}$  in others. The more precise question is then whether  $\mathbf{k}$  exists in the rest frame, same as  $\mathbf{s}$ .) Microscopically,  $\mathbf{k}$  may be conceived as arising from individual mass dipole moments of the molecules or atoms that are not compensated by shifting the coordinates of the particles, say because they are embedded in a lattice, pivoted off center.

Although the intrinsic angular momentum  $\mathbf{s}$  has frequently been included in hydrodynamic considerations, other authors see this as *ad hoc* and inconsistent, because  $\mathbf{s}$  is a relaxing variable, and its inclusion only affects high frequency, nonhydrodynamic phenomena. Redefining the momentum density, they then argue that  $\mathbf{s}$  can always be chosen to vanish, and therefore never needs to be considered [7]. While the first part of this stance is fairly convincing, the final conclusion excludes a whole category of hydrodynamic phenomena. These include especially the Einstein–de Haas effect, in which a stationary magnetizable body starts to rotate when the external magnetic field is turned off [8]: The vanishing velocity  $\mathbf{v}$  of a stationary body also compels the momentum density  $\mathbf{g} = \varrho \mathbf{v}$  and the orbital angular momentum density  $\mathbf{x} \times \mathbf{g}$  to be zero. But the total angular momentum  $\mathbf{L}$  may still be finite in the presence of a magnetic field  $\mathcal{H}$  if there is a finite intrinsic angular momentum  $\mathbf{s} \sim \mathcal{H}$ . If  $\mathcal{H}$  is turned off,  $\mathbf{s}$  vanishes, yet  $\mathbf{L}$  must remain constant. Hence the system starts rotating, as observed, to compensate.

The analogous effect for the booster would be given by aligning the microscopic mass dipoles with an external electric field, i.e.,  $\mathbf{k} \sim \mathcal{E}$ . Turning the field off gets rid of the alignment and kills  $\mathbf{k}$ . The orbital part  $M\mathbf{R}_0$  compensates by displacing the crystal.

The proper way to account for this type of effects lies in the middle ground between the above two extreme points of view—taking  $\mathbf{s}$  as either completely independent or utterly negligible. As discussed in Sec. IV, we should in fact accept

$\mathbf{s}$  as a thermodynamic variable, but not a hydrodynamic one (the precise meaning of which will become clear there). The close analogy between  $\mathbf{s}$  and  $\mathbf{k}$  makes it easy to treat the latter along the same line. One of the results, based on fairly general thermodynamic considerations, is that the booster’s analog of the Einstein–de Haas effect does not exist.

(v) Finally, a truly serendipitous result: the collective, hydrodynamic behavior of systems that spontaneously break Galilean symmetries. This subject has until now eluded clarification because the relevant conserved quantity, the booster, has been neglected. Again, the analogy between the angular momentum and the booster comes in handy here, as broken rotational symmetry is a well-understood concept [7,9]. The order parameter  $\mathbf{u}$  of the broken Galilean symmetry obeys the equation of motion,  $\partial_t \mathbf{u} + \nabla \mu = 0$ . Despite the obvious similarity to the Josephson equation [10], the Goldstone modes of broken Galilean symmetries are not propagating second sound modes, but resemble the orbital waves of nematic liquid crystals.

Unfortunately, we have nothing to say concerning possible microscopic mechanisms that realize broken Galilean symmetry. The system that spontaneously breaks this symmetry needs to display indifference with respect to the inertial system it chooses while being able to sustain a velocity gradient in equilibrium. And the relevant order parameter  $\mathbf{u}$  is a velocity difference. Such a system is admittedly hard to conceive.

## II. FRAME-INDEPENDENT THERMODYNAMICS

Let us begin by deducing the thermodynamic equilibrium conditions. Maximizing the entropy, or equivalently, minimizing the energy, while holding constant the entropy and the conserved quantities, including especially the booster  $\mathbf{B}$ , we have

$$\delta \left[ \int d^3x \varepsilon - \hat{T} \int d^3x s - \hat{\mu} \int d^3x \varrho - \hat{\mathbf{V}} \cdot \int d^3x \mathbf{g} - \hat{\mathbf{\Omega}} \cdot \int d^3x (\mathbf{x} \times \mathbf{g}) - \hat{\mathbf{A}} \cdot \int d^3x (\varrho \mathbf{x} - \mathbf{g}t) \right] = 0, \quad (7)$$

with  $\hat{T}$ ,  $\hat{\mu}$ ,  $\hat{\mathbf{V}}$ ,  $\hat{\mathbf{\Omega}}$ , and  $\hat{\mathbf{A}}$  being constant Lagrange parameters. Employing the local Gibbs relation,

$$d\varepsilon = Tds + \mu d\varrho + \mathbf{v} \cdot d\mathbf{g}, \quad (8)$$

this can be written as

$$\int d^3x [(T - \hat{T})\delta s + (\mu - \hat{\mu} - \hat{\mathbf{A}} \cdot \mathbf{x})\delta \varrho + (\mathbf{v} - \hat{\mathbf{V}} - \hat{\mathbf{\Omega}} \times \mathbf{x} + \hat{\mathbf{A}}t) \cdot \delta \mathbf{g}] = 0. \quad (9)$$

From this follow the Euler-Lagrange equations

$$T = \hat{T}, \quad \mu = \hat{\mu} + \hat{\mathbf{A}} \cdot \mathbf{x}, \quad \mathbf{v} = \hat{\mathbf{V}} + \hat{\mathbf{\Omega}} \times \mathbf{x} - \hat{\mathbf{A}}t. \quad (10)$$

Comparing the last of Eqs. (10) to the velocity field of Eq. (3), we find complete agreement with

$$\hat{\mathbf{A}} = \mathbf{\Omega} \times \dot{\mathbf{R}}, \quad \hat{\mathbf{\Omega}} = \mathbf{\Omega}, \quad \hat{\mathbf{V}} = \dot{\mathbf{R}} - \mathbf{\Omega} \times \mathbf{R}_0. \quad (11)$$

Excluding the conservation of the booster is equivalent to setting  $\hat{\mathbf{A}} = \mathbf{\Omega} \times \dot{\mathbf{R}} = 0$ , cf. Eq. (7). This leads to the results of Landau and Lifshitz, mentioned in the Introduction, correct for frames in which  $\mathbf{\Omega} \parallel \dot{\mathbf{R}}$ .

In a more deliberate approach, the input of Eq. (3) is not necessary: Varying also the time and space coordinates while minimizing the energy yields two additional terms,

$$\int d^3x [\dots + (\hat{\mathbf{\Omega}} \times \mathbf{g} - \hat{\mathbf{A}} \varrho) \cdot \delta \mathbf{x} + (\hat{\mathbf{A}} \cdot \mathbf{g}) \delta t] = 0. \quad (12)$$

Taking uniform transformations in time and space,  $\delta t$ ,  $\delta \mathbf{x} = \text{const}$ , as independent variations, we conclude  $\hat{\mathbf{A}} = \mathbf{\Omega} \times \dot{\mathbf{R}}$  and (redundantly)  $\hat{\mathbf{A}} \cdot \dot{\mathbf{R}} = 0$ .

Locally expressed, Eqs. (10) become

$$\nabla T = 0, \quad \partial_i v_j + \partial_j v_i = 0, \quad \partial_t \mathbf{v} + \nabla \mu = 0, \quad (13)$$

which are useful for the off-equilibrium considerations below. There are three corollary conclusions to draw.

(i) Integrating Eq. (8) over the system's constant volume while heeding Eqs. (10), we obtain the Galilean covariant version of the extensive, basic formula of the thermodynamics,

$$dE = \hat{T} dS + \hat{\mu} dM + \hat{\mathbf{V}} \cdot d\mathbf{G} + \hat{\mathbf{\Omega}} \cdot d\mathbf{L} + \hat{\mathbf{A}} \cdot d\mathbf{B}. \quad (14)$$

If the volume is allowed to vary, there is an additional term,  $-\oint d^2x P du$ , with  $P \equiv \mu \varrho + Ts + \mathbf{v} \cdot \mathbf{g} - \varepsilon$  and  $du$  denoting the displacement of the surface, along the surface normal and at the area element  $d^2x$ . The unusual form is related to the fact that  $P$  is not a constant, and the energy change depends on where the volume change takes place. For  $P = \text{const}$ , this term reduces to the usual form  $-P \oint d^2x du = -P dV$ . The last three terms of Eq. (14), i.e., the kinetic part of the energy, can be integrated (for a sphere) to become

$$E_{\text{kin}} = \mathbf{G}^2/2M + (\mathbf{L} - \mathbf{R}_0 \times \mathbf{G})^2/2\Theta, \quad (15)$$

as it should.  $\Theta$  is the moment of inertia in the center of mass frame.

(ii) Despite the manifest frame dependence of  $\mu$  and  $\mathbf{v}$  in Eq. (10), most quantities are properly frame independent. Take, e.g., the density distribution under rotation: The chemical potential of the local rest frame,  $\mu_0 = \mu + \frac{1}{2}v^2$ , is a function of  $T$  and  $\varrho$ , hence  $\nabla \mu_0 = (\partial \mu_0 / \partial \varrho) \nabla \varrho$  for constant temperatures. On the other hand, we deduce from Eqs. (10) and (11) that

$$\nabla \mu_0 = \nabla \left( \mu + \frac{1}{2}v^2 \right) = \frac{1}{2} \nabla [\mathbf{\Omega} \times (\mathbf{x} - \mathbf{R})]^2 \quad (16)$$

depends only on the velocity in the center of mass frame, and not on  $\dot{\mathbf{R}}$ , the center of mass velocity of a given frame. So  $\nabla \mu_0$ , and therefore  $\nabla \varrho$ , remain unchanged under a Galilean boost.

(iii) The Navier-Stokes equation is  $d\mathbf{v}/dt + \nabla P/\varrho = 0$  for  $\partial_i v_j + \partial_j v_i = 0$ , and  $d\mathbf{v}/dt + \nabla \mu_0 = 0$  if in addition  $\nabla T = 0$ . Rewriting  $d\mathbf{v}/dt \equiv \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \partial_t \mathbf{v} - \frac{1}{2} \nabla v^2$ , it finally reduces to the third of Eqs. (13),  $\partial_t \mathbf{v} + \nabla \mu = 0$ .

### III. THE DISSIPATIVE MASS CURRENT

The inclusion of the booster is clearly important for static, thermodynamic considerations, but it is equally relevant for off-equilibrium, dynamic situations: First, the proof for  $J = \mathbf{g}$ , or that no dissipative mass current is allowed. As this proof follows closely the one that deduces the symmetry of the stress tensor from local conservation of angular momentum, it aids comprehension to present both simultaneously. Rewriting the continuity equations for mass and momentum,

$$\partial_t \varrho + \partial_i J_i = 0, \quad \partial_i g_i + \partial_j \Pi_{ij} = 0, \quad (17)$$

as

$$\partial_t \ell_i + \partial_j (\epsilon_{ikm} x_k \Pi_{mj}) = \epsilon_{ijk} \Pi_{kj}, \quad (18)$$

$$\partial_t b_i + \partial_j (x_i J_j - \Pi_{ij} t) = J_i - g_i, \quad (19)$$

where  $\ell \equiv \mathbf{x} \times \mathbf{g}$  and  $\mathbf{b} \equiv \varrho \mathbf{x} - \mathbf{g}t$ , we immediately see that  $\epsilon_{ijk} \Pi_{kj}$  and  $J - \mathbf{g}$  have to vanish for the angular momentum and the booster to be locally conserved.

Now, one may argue that it would be quite enough if these two expressions can be written as divergences of some currents,  $\epsilon_{ijk} \Pi_{kj} = \partial_k \sigma_{ik}$  and  $J_i - g_i = \partial_k J_{ik}$ , so they need not vanish. But this neglects the following two points: First, qualitatively,  $\ell$  and  $\mathbf{b}$  contain the reference to the origin of the coordinate,  $\mathbf{g}$  and  $\varrho$  do not. Consider a small volume element far away from the origin, at distance  $\mathbf{R}$  and  $t = 0$ , then  $\ell$  and  $\mathbf{b} = \varrho \mathbf{x}$  will scale with  $\mathbf{R}$ . Now, as we can change  $\mathbf{R}$  by simply relocating the origin, without altering any of the physics,  $\partial_t \ell$ ,  $\partial_t \mathbf{b}$ , and their respective fluxes must also scale with  $\mathbf{R}$ . On the other hand, because  $\Pi_{ij}$  and  $J$  (as fluxes of  $\mathbf{g}$  and  $\varrho$ ) do not, neither do  $\epsilon_{ijk} \Pi_{kj} = \partial_k \sigma_{ik}$  and  $J_i - g_i = \partial_k J_{ik}$ , which therefore must vanish.

The second point is more quantitative: Being currents of hydrodynamic variables,  $\Pi_{ij}$  and  $J_i$  are themselves functions of these variables and their spatial derivatives. To lowest order, with no spatial derivatives,  $\Pi_{ij}$  and  $J_i$  cannot possibly be written as divergences of some currents. In the next order, the terms are mostly dissipative. Taking  $\epsilon_{ijk} \Pi_{kj}$ ,  $J - \mathbf{g}$  as finite, we find the entropy production to be given as [4,11]

$$\mathcal{R} = \dots + \epsilon_{ijk} \Pi_{jk} \Omega_i - (J - \mathbf{g}) \cdot \nabla \mu, \quad \mathbf{\Omega} \equiv \frac{1}{2} \nabla \times \mathbf{v}. \quad (20)$$

According to the rules of irreversible thermodynamics, if the entropy production is given as a sum of products,  $\mathcal{R} = \sum X_i Y_i$ , we may take the fluxes  $X_i$  as proportional to the thermodynamic forces  $Y_i$  [11], hence

$$\epsilon_{ijk} \Pi_{jk} = \zeta_{(1)} \Omega_i, \quad J - \mathbf{g} = -\zeta_{(2)} \nabla \mu, \quad (21)$$

where  $\zeta_{(1)}$  and  $\zeta_{(2)}$  are transport coefficients, similar to the viscosity. (More accurately, the  $X_i$  are a linear combination of the  $Y_i$ , so off-diagonal, cross terms are also possible. However, these terms always vanish if the diagonal ones do, so they need no extra consideration.) Equations (21) lead to the entropy production

$$\mathcal{R} = \dots + \zeta_{(1)} \mathbf{\Omega}^2 + \zeta_{(2)} (\nabla \mu)^2, \quad (22)$$

which, however, contradicts Eqs. (13):  $\boldsymbol{\Omega}$  and  $\nabla\boldsymbol{\mu}$  need not vanish even if all the three fields there do, in which case equilibrium reigns with  $\mathcal{R}\equiv 0$ . Therefore  $\zeta_{(1)}$ ,  $\zeta_{(2)}$  are zero, and  $\epsilon_{ijk}\Pi_{jk}$ ,  $J-\mathbf{g}$  vanish.

#### IV. INTRINSIC BOOSTER DENSITY

Allowing now intrinsic contributions, as in Eqs. (5) and (6), the energy density depends on two additional variables,  $\mathbf{s}$  and  $\mathbf{k}$ , or

$$d\varepsilon = Tds + \mu d\rho + \mathbf{v}\cdot d\mathbf{g} + \boldsymbol{\omega}\cdot d\mathbf{s} + \mathbf{a}\cdot d\mathbf{k}. \quad (23)$$

Barring an instability, the energy  $\varepsilon$  is minimal for  $\mathbf{s}$ ,  $\mathbf{k}=\mathbf{0}$ . Therefore an expansion around the minimal value  $\varepsilon_{\min}$  would usually yield a quadratic dependence,  $\varepsilon = \varepsilon_{\min} + \frac{1}{2}(\gamma_{(1)}\mathbf{s}^2 + \gamma_{(2)}\mathbf{k}^2)$ . (Isotropy is assumed, otherwise  $\gamma_{(1)}$ ,  $\gamma_{(2)}$  are tensors rather than scalars.) Being derivatives,  $\boldsymbol{\omega}$  and  $\mathbf{a}$  assume the form

$$\boldsymbol{\omega} = \gamma_{(1)}\mathbf{s}, \quad \mathbf{a} = \gamma_{(2)}\mathbf{k}. \quad (24)$$

If the magnetic field is included,  $d\varepsilon = \dots + \mathcal{H}\cdot d\mathcal{B}$ , the energy is minimal for  $\mathbf{s}$ ,  $\mathbf{k}$ ,  $\mathcal{B}=\mathbf{0}$ , and the same expansion (again for an isotropic medium) yields

$$\boldsymbol{\omega} = \gamma_{(1)}\mathbf{s} + \gamma_{(3)}\mathcal{B}. \quad (25)$$

It is this cross dependency that leads to the Einstein–de Haas effect discussed in the Introduction [8]. Similarly, an electric field leads to

$$\mathbf{a} = \gamma_{(2)}\mathbf{k} + \gamma_{(4)}\mathcal{D}. \quad (26)$$

Because  $\mathcal{L}+\mathbf{s}$  and  $\mathbf{b}+\mathbf{k}$  are locally conserved quantities and satisfy continuity equations, the new variables obey the equations of motion,

$$\partial_t s_i + \partial_j \lambda_{ij} = \epsilon_{ijk}\Pi_{jk}, \quad \partial_t k_i + \partial_j \gamma_{ij} = g_i - J_i, \quad (27)$$

where  $\lambda_{ij}$  and  $\gamma_{ij}$  are the respective currents, functions of the local hydrodynamic variables, the explicit form of which we shall not derive here. The source terms on the right hand side cancel those of Eqs. (19), and ensure that  $\mathcal{L}+\mathbf{s}$  and  $\mathbf{b}+\mathbf{k}$  satisfy continuity equations. (The vanishing of  $\epsilon_{ijk}\Pi_{jk}$  and  $\mathbf{g}-J$  was deduced in the preceding section as a direct consequence of  $\mathcal{L}$  and  $\mathbf{b}$  being locally conserved quantities. As this is no longer true, we have no prior knowledge of whether  $\epsilon_{ijk}\Pi_{jk}$  and  $\mathbf{g}-J$  are finite.) Going through the same hydrodynamic procedure as that leading to Eq. (20), we find the modified entropy production,

$$\mathcal{R} = \dots + \epsilon_{ijk}\Pi_{jk}(\Omega_i - \omega_i) - (J-\mathbf{g})\cdot(\nabla\boldsymbol{\mu}-\mathbf{a}), \quad (28)$$

and conclude as before,

$$\epsilon_{ijk}\Pi_{jk} = \zeta_{(1)}(\Omega_i - \omega_i), \quad J-\mathbf{g} = -\zeta_{(2)}(\nabla\boldsymbol{\mu}-\mathbf{a}), \quad (29)$$

If the two coefficients were exceedingly large,  $\zeta_{(1)}$ ,  $\zeta_{(2)} \rightarrow \infty$ , we may divide each of Eqs. (27) by the respective coefficient and find the equations substituted by the two constraints,

$$\boldsymbol{\omega} = \boldsymbol{\Omega}, \quad \mathbf{a} = \nabla\boldsymbol{\mu}. \quad (30)$$

The first implies instantaneous establishment of partial equilibrium by local exchanges of angular momentum, between the reservoirs  $\mathcal{L}$  and  $\mathbf{s}$ ; the second implies partial equilibrium by local exchanges of booster, between  $\mathbf{b}$  and  $\mathbf{k}$ . So instead of an independent dynamics,  $\mathbf{s}$  and  $\mathbf{k}$  assume the appropriate values satisfying Eqs. (30), instantaneously during hydrodynamic processes.

For generic circumstances, this is indeed the case. There are two terms in Eqs. (29),

$$\zeta_{(1)}\boldsymbol{\omega} = \zeta_{(1)}\gamma_{(1)}\mathbf{s} \equiv \mathbf{s}/\tau_{(1)}, \quad \zeta_{(2)}\mathbf{a} = \zeta_{(2)}\gamma_{(2)}\mathbf{k} \equiv \mathbf{k}/\tau_{(2)},$$

which show that both Eqs. (27) are relaxation equations, and that Eqs. (30) hold after the time  $\tau_{(1)}$  and  $\tau_{(2)}$ , respectively. Generally speaking, these characteristic times, after which local equilibrium is established, are microscopic in nature and much faster than typical hydrodynamic time scales. And taking these times as negligibly small is equivalent to the limit  $\zeta_{(1)}$ ,  $\zeta_{(2)} \rightarrow \infty$ , and equivalent to substituting Eqs. (30) for the equations of motion (27). Note, however, that this dynamic dependency does not at all mean  $\mathbf{s}$  and  $\mathbf{k}$  cannot remain full-fledged thermodynamic variables, as Eqs. (23)–(26) clearly remain valid.

From a more elevated point of view, we understand that the universality and simplicity of hydrodynamic theories are achieved by eliminating all relaxing variables, and expunging all relaxation equations. As a result, the hydrodynamic theory is confined to frequencies much less than all relaxation rates  $1/\tau_{(i)}$ .

(A possible source of confusion is the seemingly odd fact that the transport coefficients  $\zeta_{(1)}$  and  $\zeta_{(2)}$  were deduced to be zero in the preceding section, yet argued to be diverging here. The explanation lies in the difference between the calibers of the arguments: In the last section, the coefficients were not negligibly small. Rather, they had to vanish identically to not contradict thermodynamics. Here, the coefficients are finite, but well approximated by  $\infty$  when compared to hydrodynamic time scales. In other words, if they are finite, they are very large—and there is no contradiction.)

One problem remains: The two terms  $\epsilon_{ijk}\Pi_{jk}$  and  $J-\mathbf{g}$  are now indeterminate, as they are both products of an infinite and a vanishing factor, cf. Eqs. (29). Nevertheless, they need to be known before we may make use of the continuity equations for mass and momentum, Eqs. (17). A clever way to accomplish this is to go over to the new densities [7],

$$\tilde{\mathbf{g}} = \mathbf{g} + \frac{1}{2}(\nabla \times \mathbf{s}), \quad \tilde{\varrho} = \varrho - \nabla \cdot \mathbf{k}. \quad (31)$$

They are obtained by starting from Eq. (23), with Eqs. (30) incorporated,

$$d\varepsilon = \dots + \mathbf{v}\cdot d\mathbf{g} + \boldsymbol{\Omega}\cdot d\mathbf{s} + \mu d\rho + \nabla\boldsymbol{\mu}\cdot d\mathbf{k}. \quad (32)$$

This is partially integrated to yield

$$d\varepsilon = \dots + \mathbf{v}\cdot d\tilde{\mathbf{g}} + \mu d\tilde{\varrho}, \quad (33)$$

which shows first of all that the energy density is well accounted for by the new densities. Putting the surface slightly beyond the system's volume, we also have

$$\int d^3x \tilde{\mathbf{g}} = \int d^3x \mathbf{g}, \quad \int d^3x \tilde{\varrho} = \int d^3x \varrho, \quad (34)$$

as the respective second terms of Eq. (31) are surface terms. So we conclude that  $\tilde{\mathbf{g}}$  and  $\tilde{\varrho}$  are also valid densities for momentum and mass. Most crucially, because of

$$\int d^3x (\mathbf{x} \times \tilde{\mathbf{g}}) = \int d^3x (\mathbf{x} \times \mathbf{g} + \mathbf{s}), \quad (35)$$

and

$$\int d^3x (\tilde{\varrho} \mathbf{x} - \tilde{\mathbf{g}} t) = \int d^3x (\varrho \mathbf{x} - \mathbf{g} t + \mathbf{k}), \quad (36)$$

these new densities clearly absorb the intrinsic contributions,  $\mathbf{s}$ ,  $\mathbf{k}$ , and therefore have fluxes for which

$$\epsilon_{ijk} \tilde{\Pi}_{jk} = 0, \quad \tilde{\mathbf{J}} - \tilde{\mathbf{g}} = 0 \quad (37)$$

hold. So it is indeed a good idea to take  $\tilde{\varrho}$  and  $\tilde{\mathbf{g}}$  as the hydrodynamic variables. (But it is now also possible to return to the original variables: Given  $\tilde{\Pi}_{ij}$  and  $\tilde{\mathbf{J}}$ , it is easy to obtain  $\Pi_{ij} = \tilde{\Pi}_{ij} + \frac{1}{2} \epsilon_{ijk} \partial_r s_k$ ,  $\mathbf{J} = \tilde{\mathbf{J}} - \partial_t \mathbf{k}$ .)

All this does not mean that no experimental consequences are related to  $\mathbf{s}$  and  $\mathbf{k}$ , as claimed in [7]. For instance, the constitutive relations,  $\mathbf{g} = \varrho \mathbf{v}$  and Eq. (25), or

$$\tilde{\mathbf{g}} = \varrho \mathbf{v} + \frac{1}{2} \nabla \times \mathbf{s}(\boldsymbol{\Omega}, \boldsymbol{\mathcal{B}}), \quad (38)$$

cannot be without consequences in the Navier-Stokes equation,  $\partial_t \tilde{\mathbf{g}}_i = \tilde{\Pi}_{ij}$ : A temporally varying but spatially homogeneous  $\boldsymbol{\mathcal{B}}$  field will lead to an acceleration,  $\partial_t(\varrho \mathbf{v}) = -\partial_t[\nabla \times \mathbf{s}(\boldsymbol{\mathcal{B}})/2]$ , at the system's outer rim, the only place where  $\nabla \times \mathbf{s}$  is nonzero. The resultant rotary motion there then propagates into the bulk, elastically in a solid and viscously in a liquid. Clearly, this is a temporally resolved, hydrodynamic description of the Einstein–de Haas effect.

Other thermodynamic cross dependencies of  $\mathbf{s}$  will lead to similar behavior. The diagonal term all by itself, however, while necessary for maintaining the thermodynamic stability, is probably always negligibly small, as

$$\nabla \times \mathbf{s}/2 = \nabla \times \boldsymbol{\Omega}/2\gamma_{(1)}$$

is of the same order as the viscous terms.

In spite of the one-to-one analogy,

$$\partial_t \tilde{\varrho} = \partial_t[\varrho - \nabla \cdot \mathbf{k}]$$

instead of Eq. (38), there is one crucial difference between  $\mathbf{s}$  and  $\mathbf{k}$ : The quantity

$$\partial \varepsilon / \partial \mathbf{k} = \nabla \mu = \boldsymbol{\Omega} \times \dot{\mathbf{R}}$$

depends on the inertial frame. As a result, thermodynamic cross dependencies are ruled out for  $\mathbf{k}$ . To understand why, consider  $\mathbf{k}(\boldsymbol{\mathcal{E}})$ , the electric analog of  $\mathbf{s}(\boldsymbol{\mathcal{B}})$ . Due to the Maxwell relation

$$\partial k_j / \partial \mathcal{E}_i = \partial \mathcal{D}_i / \partial (\partial_j \mu) \quad (39)$$

that follows from the energy  $d\varepsilon = \dots + \mathcal{E}_i d\mathcal{D}_i + (\partial_j \mu) dk_j$ ,  $\mathbf{k}$ 's dependence on  $\boldsymbol{\mathcal{E}}$  implies  $\mathcal{D}$ 's on  $\nabla \mu$ , a quantity that can be made to vanish by choosing an appropriate frame. In the absence of a magnetic field,  $\mathcal{D}$  is essentially frame independent,  $\mathcal{D}(\nabla \mu) = \mathcal{D}(0)$ , or  $\partial \mathcal{D}_i / \partial (\partial_j \mu) = 0$ . Hence  $\partial k_j / \partial \mathcal{E}_i = 0$ .

The diagonal term  $\gamma_{(2)} \mathbf{k} = \nabla \mu = \boldsymbol{\Omega} \times \dot{\mathbf{R}}$  remains, but the associated hydrodynamic effects,  $\partial_t \nabla \cdot \mathbf{k} \sim (\partial \mu / \partial \varrho) \partial_t \nabla^2 \varrho$ , are again an order smaller than the dissipative terms. They may probably always be ignored. Nevertheless, understanding the treatment of a finite  $\mathbf{k}$  is useful, both for its own sake and for the consideration below of broken Galilean symmetry, (although these systems do not necessarily have a finite  $\mathbf{k}$ .)

## V. SPONTANEOUSLY BROKEN GALILEAN SYMMETRY

Thermodynamic and hydrodynamic theories account for the generic behavior of macroscopic systems that results from conservation laws and broken symmetries. Although the specificity of its predictive power is, compared to the general input, frequently amazing, some subjects are simply beyond its reach. One example is the question concerning possible microscopic mechanisms which lead to broken Galilean symmetries, or in fact whether it at all exists. Fortunately, we do not need this knowledge to give a full account of the collective behavior of a system that breaks this symmetry, and hereby filling the macroscopic half of a large gap in one of our more basic concepts.

Starting with a system that breaks rotational invariance in all three directions (say biaxial nematics or the  $B$  phase of superfluid  $^3\text{He}$  [12]) we take the infinitesimal rotation angle  $d\theta_i$  as the order parameter, with the energy depending on its gradient,

$$d\varepsilon = \dots + \Omega_i ds_i + \phi_{ij} d(\partial_j \theta_i). \quad (40)$$

An expansion of the energy  $\varepsilon$  leads to  $\phi_{kl} \sim \partial_j \theta_i$ . The equations of motion in the rest frame possess the usual Hamiltonian form,

$$\partial_t \theta_i = \delta \varepsilon / \delta s_i = \Omega_i, \quad \partial_t s_i = -\delta \varepsilon / \delta \theta_i = \partial_j \phi_{ij}. \quad (41)$$

Eliminating  $\partial_t s_i$  as outlined in the preceding section, the continuity equation for the momentum gets modified:

$$\partial_t \tilde{\mathbf{g}}_i = \partial_t g_i + \frac{1}{2} \epsilon_{ijk} \partial_j \partial_r s_k = \frac{1}{2} \epsilon_{ijk} \partial_j \partial_m \phi_{km} + \dots,$$

which leads to three pairs of orbital waves, of the form  $\omega \sim q^2$ , with complex coefficients [12]. (A slight rearrangement is needed to render the seemingly antisymmetric stress tensor symmetric.)

The order parameter of broken Galilean symmetry is a velocity field,  $u_i$ , with the energy depending on its gradient,

$$d\varepsilon = \dots + (\partial_i \mu) dk_i + J_{ij} d(\partial_j u_i). \quad (42)$$

An expansion leads to  $J_{kl} \sim \partial_j u_i$ . The equations of motion in the rest frame again possess the usual Hamiltonian form,

$$\partial_t u_i = -\delta \varepsilon / \delta k_i = -\partial_i \mu, \quad \partial_t k_i = \delta \varepsilon / \delta u_i = -\partial_j J_{ij}. \quad (43)$$

Eliminating  $\partial_i k_i$  as outlined in the preceding section, the continuity equation for the mass density gets modified:

$$\partial_t \tilde{\rho} = \partial_t \rho - \partial_i \partial_t k_i = -\partial_i (\rho v_i - \partial_j J_{ij}),$$

where the total mass current equals the momentum density. Including nonlinear terms and the diagonal dissipative one, we have (dropping the tilde)

$$\partial_t u_i + (v_k \partial_k) u_i + \partial_i \mu - \zeta \partial_j J_{ij} = 0, \quad (44)$$

$$\partial_t \rho + \partial_i g_i = 0, \quad g_i = \rho v_i - \partial_j J_{ij}. \quad (45)$$

In conjunction with the rest of the hydrodynamic equations that essentially retain their form from the isotropic liquid, we find the sound mode to be unchanged, while three additional modes, of coupled temperature and  $u$  motion, are (as in the orbital case) of the form  $\omega \sim q^2$ , with the coefficients being complex.

## VI. SUMMARY

The three locally conserved densities that result from a system's invariance under Galilean transformation have been included as variables in the thermodynamic and hydrodynamic theory. Utilizing their close analogy to the angular momentum, the thermodynamics is rendered explicitly frame independent, and dissipative mass currents are shown to vanish identically. In addition, two corollary results have been obtained: First, in contrast to the useful concept of an intrinsic angular momentum density, an intrinsic booster density is shown to be one that we most probably need not embrace; and two, the collective behavior of systems that spontaneously break Galilean symmetry have been derived.

## ACKNOWLEDGMENT

It was exactly 20 years ago, while we were scrambling to decipher the many broken symmetries of superfluid  $^1\text{He}$ , when Grisha Volovik insisted that broken Galilean symmetry must be a sensible concept. I did not believe him then, but I am happy now that he was right.

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